A. Kaestner :: Paul Scherrer Institut

Image enhancement

Preparing the images for analysis

PAUL SCHERRER INSTITUT

Outline

1 Introduction

2 Noise and Artifacts

3 Basic filtering

4 Scale spaces

5 Verification

6 Summary

Introduction

Introduction

3D and 4D imaging produce large amounts of data

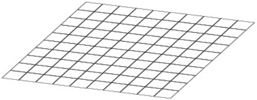
Gigabytes. . .
. . . or even
terabytes of data

- 3D visualization
- Sample characteriztation
- Process parameterization
- etc

Different types of images

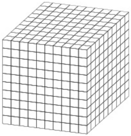
2D

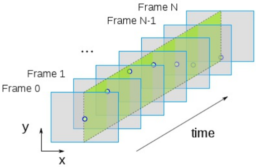
- Pictures
- Radiographs
- CT slices



3D

- Volumes
  x, y, z
- Movies
  x, y, t





4D

- Volume movie
  x, y, z, t

Which information do you want to gain

Quantitative

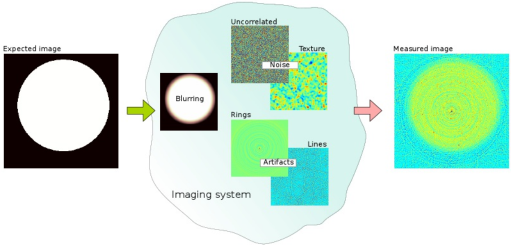
- Material composition
- Material transport

Structure

- Identify items
- Item geometry

This will affect the choice of processing methods.

Measurements are rarely perfect

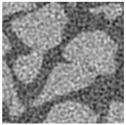


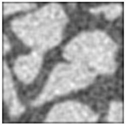
Factors affecting the image quality

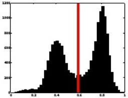
- Resolution (Imaging system transfer functions)
- Noise
- Contrast
- Inhomogeneous contrast
- Small relevant features
- Artifacts

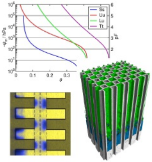
A typical processing chain

Image Acquisition	Enhancement	Segmentation	Post processing	Evaluation









Todays lecture will focus on the enhancement

Noise and artifacts

The unwanted information

Noise types

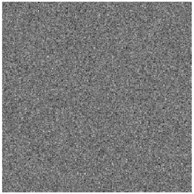
Spatially uncorrelated noise

Event noise

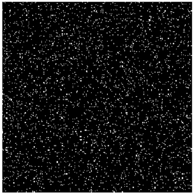
Structured noise

Noise expamles

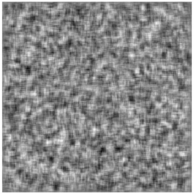
Gaussian



Salt 'n pepper



Structured



Noise models – Distributions

Gaussian noise

- Additive
- Easy to model
- Other distributions obtain Gaussian shape at large numbers

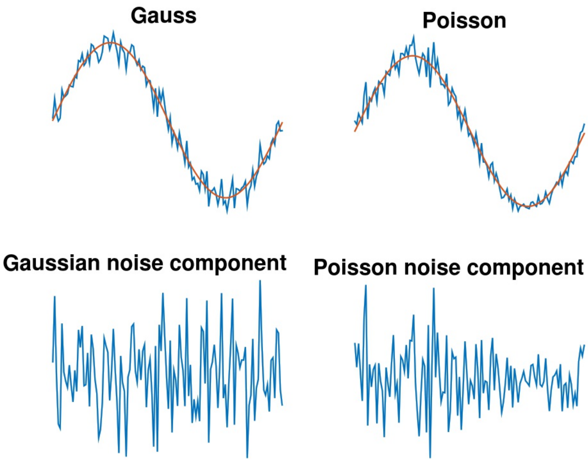
$$n(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\left(\frac{x-\mu}{2\sigma}\right)^2}$$

Poisson noise

- Multiplicative
- Physically correct for event counting

$$p(x) = \frac{\lambda^k}{k!} e^{-\lambda x}$$

Noise examples



Noise models – Salt'n'Pepper noise

- A type of outlier noise
- Noise strength give as the probability of an outlier
- Additive, multiplicative, independent replacement

Example

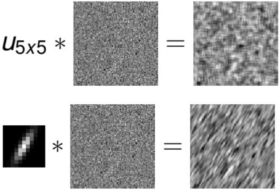
$$sp(x) = \begin{cases} -1 & x \leq \lambda_1 \\ 0 & \lambda_1 < x \leq \lambda_2 \\ 1 & \lambda_2 < x \end{cases} \qquad \begin{array}{l} x \in \mathcal{U}(0,1) \\ \lambda_1 < \lambda_2 \\ \lambda_1 + \lambda_2 = \text{noise fraction} \end{array}$$

Noise models – Structured noise

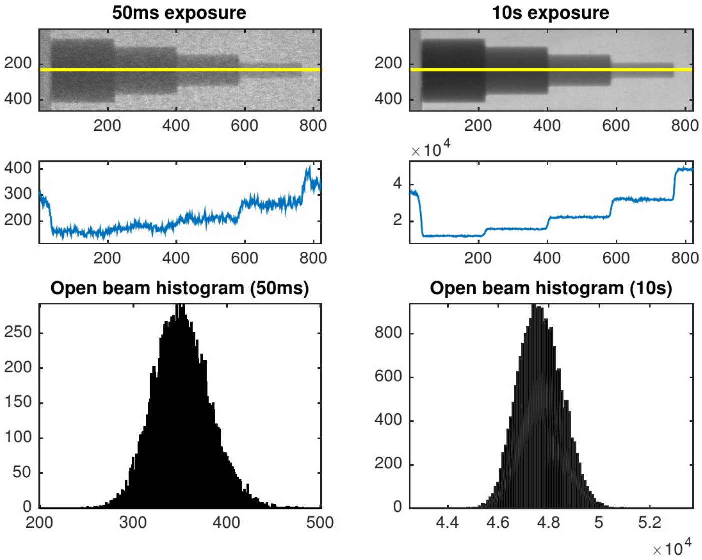
- Spatially correlated
- Example: Detector structure

Example random field models

$$
\begin{aligned}
n(x,y) &\in \mathcal{N}(\mu,\sigma) \\
ns &= K * n \qquad K = \text{convolution kernel}
\end{aligned}
$$



Noisy profiles



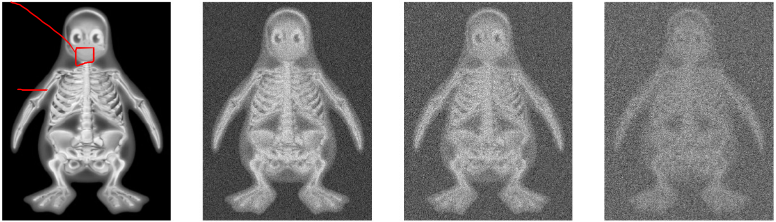
Signal to noise ratio

A metric to describe noise strength

$$SNR = \frac{\mu_{image}}{\sigma_{image}} \qquad (1)$$

$$SNR_{db} = 20 \log \frac{\mu_{image}}{\sigma_{image}} \qquad (2)$$

- Select a region
- Compute average intensity
- Compute std deviation
- Apply eqns 1 or 2



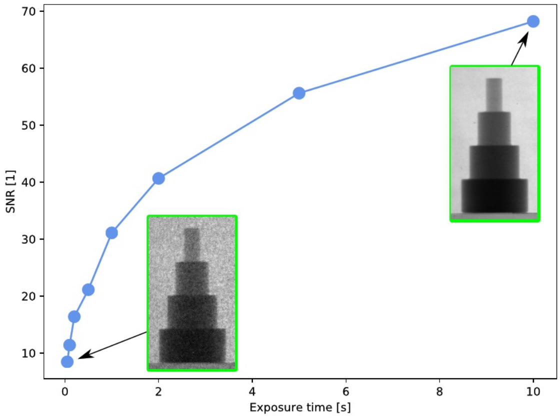
SNR = ∞ | SNR = 5 | SNR = 2 | SNR = 1

SNR for different exposure times

Poisson noise

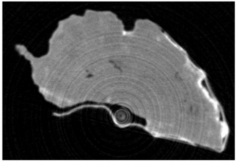
- $E[Poi(\lambda)] = \lambda$ and $var[Poi(\lambda)] = \lambda \rightarrow$ SNR=$\sqrt{\lambda}$
- Exposure time increase the number of events

$\frac{\lambda}{\sqrt{\lambda}} = \sqrt{\lambda}$



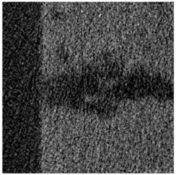
Rings

- Appear in most CT acquisitions
- Caused by stuck pixels in the projection data
- Can mostly be supressed during reconstruction



Lines

- Frequent in neutron CT slices
- Caused by spots on single projections
- Can mostly be supressed during reconstruction



Numerical artifacts

Rounding errors

- May appear with sum operations on large data sets.
- At some point the new term is smaller than the precision.

Instable processing

- Due to incorrect regularization
- Wrong parameterization
- Incorrect implementation. . . bugs *etc*

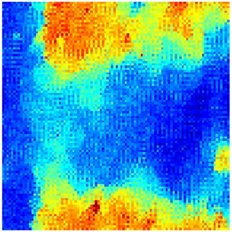


Image processing with Python

Python provides several packages for matrix handling and image processing:

numpy	Numeric operations scalars, lists, and matrices.
skimage	Image processing for 2D and 3D images.
matplotlib	Plotting and visualization.

Some relevant python functions

Random number generators [numpy.random]

Gauss	np.random.normal(μ,σ, size=[*rows*,*cols*])
Uniform	np.random.uniform(*low*,*high*,size=[*rows*,*cols*])
Poisson	np.random.poisson(λ, size=[*rows*,*cols*]) alt np.random.poisson(img, size=[*rows*,*cols*])

Generates an $m \times n$ random fields with different distributions.

Statistics

- np.mean(*f*), np.var(*f*), np.std(*f*) Computes the mean, variance, and standard deviation of an image *f*.
- np.min(*f*),np.max(*f*) Finds minimum and maximum values in *f*.
- np.median(*f*), np.rank()

Basic filtering

The first approach to image enhancement

General definition

A filter is a processing unit that

- Enhances the wanted information
- Suppresses the unwanted information

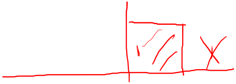
Ideally without altering relevant features beyond recognition

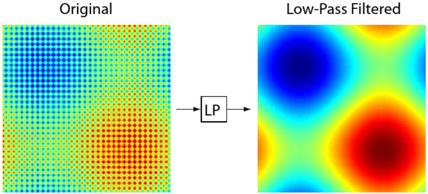
[Jähne, 2005]

Filter characteristics

Low pass filters

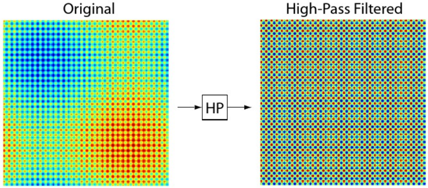
- Slow changes are enhanced
- Rapid changes are suppressed





High pass filters

- Rapid changes are enhanced
- Slow changes are suppressed

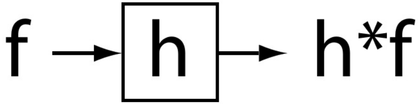


Computed using the convolution operation

$$g(\mathbf{x}) = h * f(\mathbf{x}) = \int_{\Omega} f(\mathbf{x} - \boldsymbol{\tau})\, h(\boldsymbol{\tau})\, d\boldsymbol{\tau} \quad (3)$$

where

- $f(\mathbf{x})$ is the image
- h is the convolution kernel of the filter



Low-pass filter kernels

Mean or Box filter

All weights have the same value.

Example:

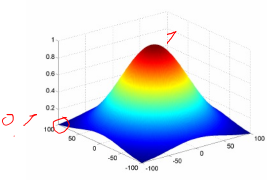
$$B = \frac{1}{25} \cdot$$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

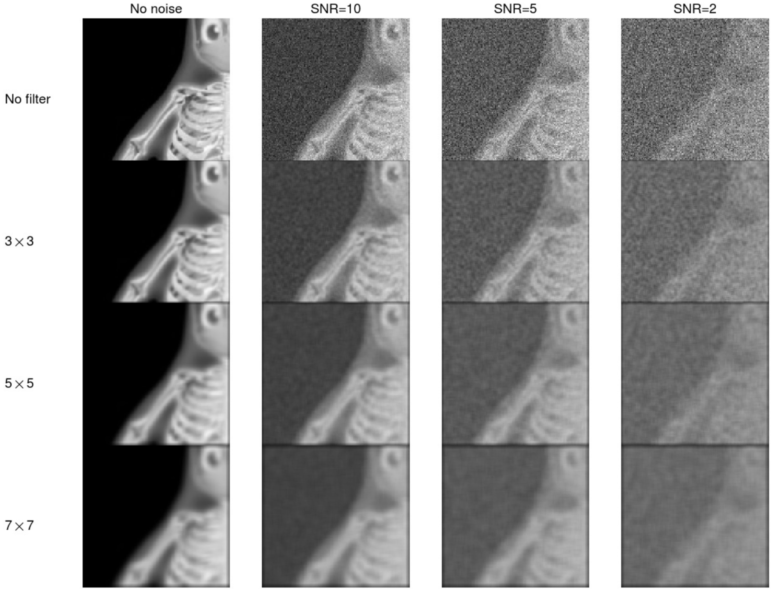
Gauss filter

$$G = e^{-\frac{x^2+y^2}{2\,\sigma^2}}$$

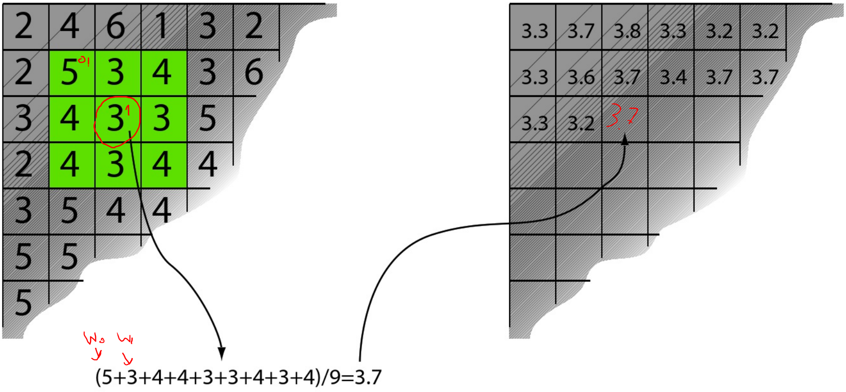
Example:



Using a Mean filter



How is the convolution computed



Note

For a non-uniform kernel each term is weighted by its kernel weight.

Euclidean separability

The asociative and commutative laws apply to convoution

$$(a * b) * c = a * (b * c) \quad \text{and} \quad a * b = b * a$$

A convolution kernel is called *separable* if it can be slit in two or more parts:

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} * \begin{bmatrix} \cdot & \cdot & \cdot \end{bmatrix}$$

Gain

Reduces the number of computations $\rightarrow$ faster processing

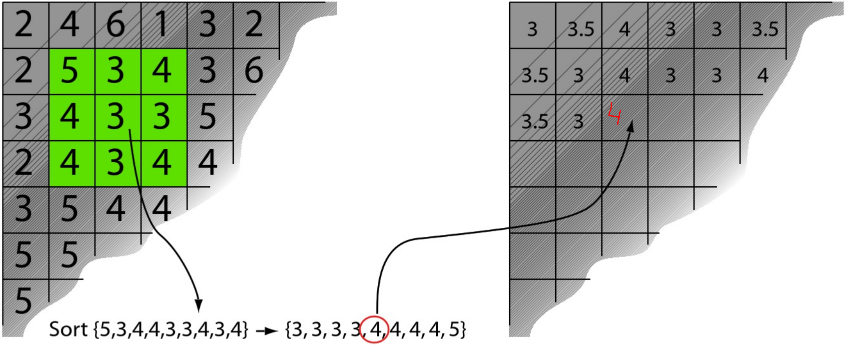
$$3\times3 \rightarrow 9 \text{ mult and } 8 \text{ add} \Leftrightarrow 6 \text{ mult and } 4 \text{ add}$$

$$3\times3\times3 \rightarrow 27 \text{ mult and } 26 \text{ add} \Leftrightarrow 9 \text{ mult and } 6 \text{ add}$$

Example

$$e^{-\frac{x^2+y^2}{2\,\sigma^2}} = e^{-\frac{x^2}{2\,\sigma^2}} * e^{-\frac{y^2}{2\,\sigma^2}}$$

The median filter



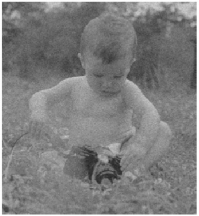
Comparing filters for different noise types

10% salt&pepper noise

Median filtered

Mean filtered

Additive White Gaussian noise (σ =30)



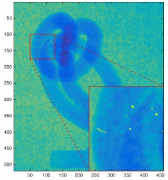
Median filtered

Mean filtered

Filter example: Spot cleaning

Problem

- Many neutron images are corrupted by spots that confuse following processing steps.
- The amount, size, and intensity varies with many factors.



Solutions

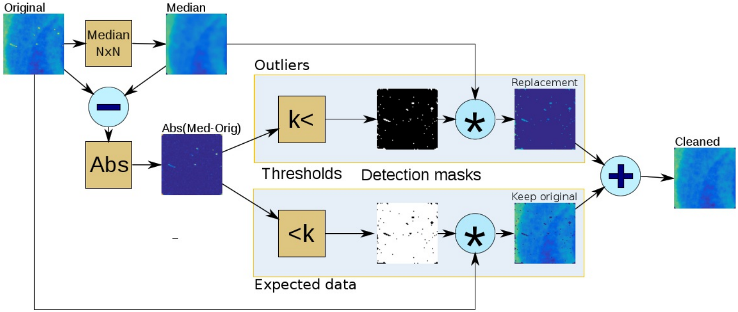
:-(Low pass filter

:-(Median filter

:-) Detect spots and replace by estimate

Spot cleaning algorithm

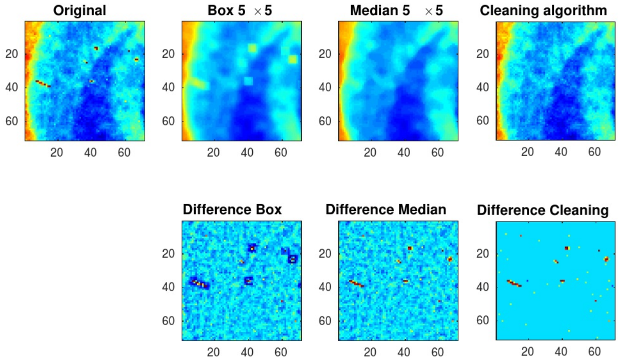
An algorithm



Parameters

- *N* Width of median filter.
- *k* Threshold level for outlier detection.

Spot cleaning – Compare performance



The ImageJ ways

Despeckle Median ... please avoid this one!!!

Remove outliers Similar to cleaning described algorithm

High-pass filters

High-pass filters enhance rapid changes – ideal for edge detection

Typical high-pass filters:

Gradients

−1 0 1

$$\frac{\partial}{\partial x} = \frac{1}{2} \cdot \begin{array}{|c|c|}\hline -1 & 1 \\ \hline\end{array}$$

$$\frac{\partial}{\partial x} = \frac{1}{32} \cdot \begin{array}{|c|c|c|}\hline -3 & 0 & 3 \\ \hline -10 & 0 & 10 \\ \hline -3 & 0 & 3 \\ \hline\end{array}$$

Laplacian

$$\triangle = \frac{1}{2} \cdot \begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 2 & -12 & 2 \\ \hline 1 & 2 & 1 \\ \hline\end{array}$$

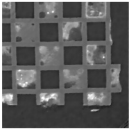
Sobel

$$G = |\nabla f| = \sqrt{\left(\frac{\partial}{\partial x} f\right)^2 + \left(\frac{\partial}{\partial y} f\right)^2}$$

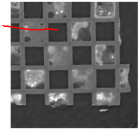
Gradient example

Vertical edges

$$\frac{\partial}{\partial x}$$



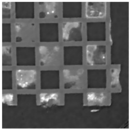
$$= \frac{1}{32} \cdot \begin{bmatrix} -3 & 0 & 3 \\ -10 & 0 & 10 \\ -3 & 0 & 3 \end{bmatrix} *$$



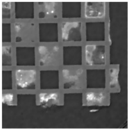
$=$

Horizontal edges

$$\frac{\partial}{\partial y}$$



$$= \frac{1}{32} \cdot \begin{bmatrix} -3 & -10 & -3 \\ 0 & 0 & 0 \\ 3 & 10 & 3 \end{bmatrix} *$$



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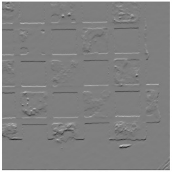
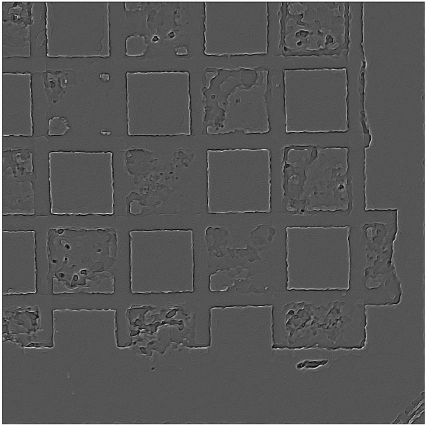


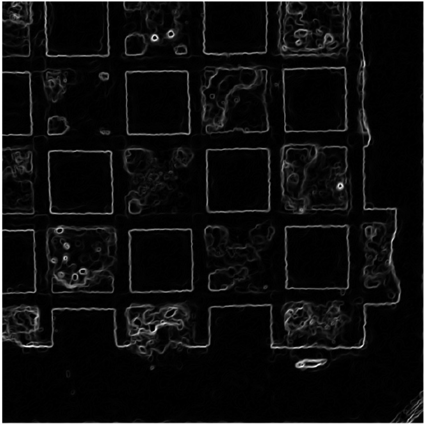
Edge detection examples

Laplacian



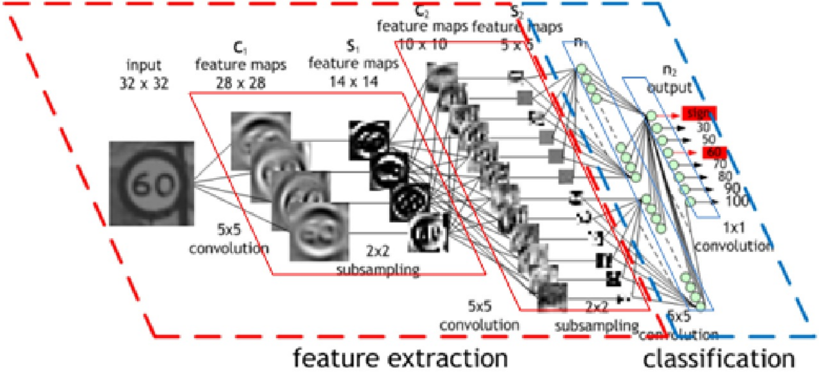
Both negative and positive values

Sobel



Positive values only.

Relevance to machine learning



NVIDIA Developer zone

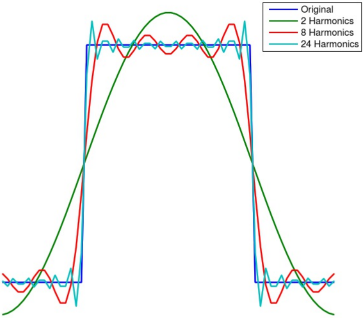
Filters in frequency space

Applications of the Fourier transform

What is frequency space?

Introduction

- A signal can be decomposed into a sum of basic harmonics defined by amplitude, phase shift and frequency.
- Fine details and sharp edges require more harmonics



The Fourier transform

Transform

$$G(\xi_1,\xi_2)=\mathcal{F}\{g\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-i(\xi_1\,x+\xi_2\,y)}\,dx\,dy$$

It's inverse

$$g(x,y)=\mathcal{F}^{-1}\{G\}=\frac{1}{(2\,\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\omega)e^{i(\xi_1\,x+\xi_2\,y)}\,d\xi_1\,d\xi_2$$

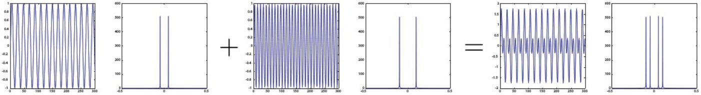
FFT

In practice – you never see the transform equations.
The Fast Fourier Transform is available in numerical libraries and tools.

[Jähne, 2005]

Addition

$$\mathcal{F}\{a+b\} = \mathcal{F}\{a\} + \mathcal{F}\{b\}$$



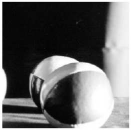
Convolution

$$\boxed{\mathcal{F}\{a * b\} = \mathcal{F}\{a\} \cdot \mathcal{F}\{b\}}$$

$$\mathcal{F}\{a \cdot b\} = \mathcal{F}\{a\} * \mathcal{F}\{b\}$$

Additive noise in Fourier space

$f + AWGN =$



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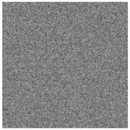

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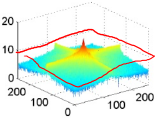
$\mathcal{F}\{f + AWGN\} =$



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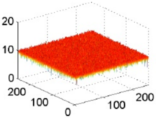

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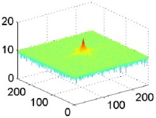
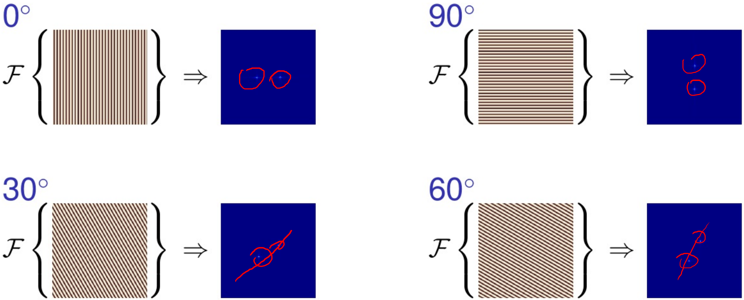


Problem

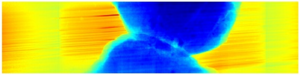
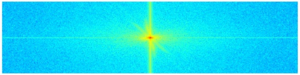
How can we suppress noise without destroying relevant image features?

Spatial frequencies and orientation

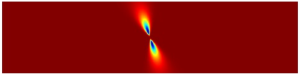
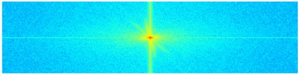


Example – Stripe removal in Fourier space

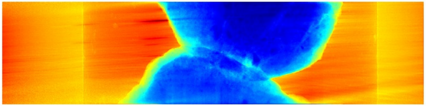
1. Transform the image to Fourier space

$\mathcal{F}\Big\{$  $\Big\} \Rightarrow$ 

2. Multiply spectrum image by band pass filter

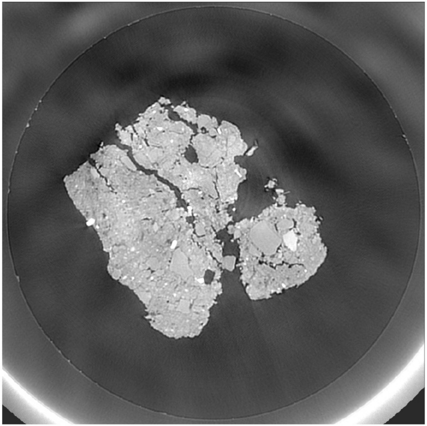
 $\cdot$ 

3. Compute the inverse transform to obtain the filtered image in real space

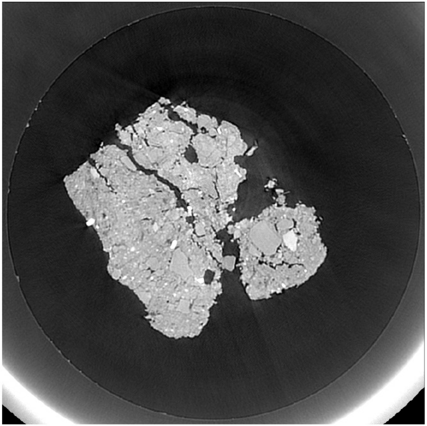


The effect of the stripe filter

Reconstructed CT slice before filter



Reconstructed CT slice after filter



Intensity variations are suppressed using the stripe filter on all projections.

Python functions

Filters in the spatial domain

e.g. from scipy import ndimage

ndimage.filters.convolve(f,h) Linear filter using kernel h on image f.

ndimage.filters.median_filter(f,n,m) Median filter using an $n \times m$ filter neighborhood

Fourier transform

np.fft.fft2(f) Computes the 2D Fast Fourier Transform of image f

np.fft.ifft2(F) Computes the inverse Fast Fourier Transform F.

Complex numbers

np.abs(f), np.angle(f) Computes amplitude and argument of a complex number.

np.real(f), np.imag(f) Gives the real and imaginary parts of a complex number.

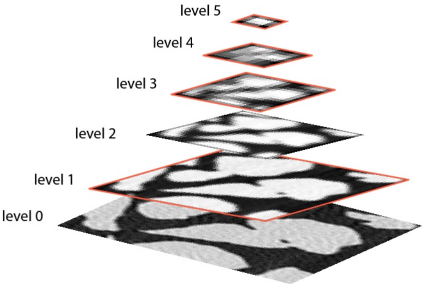
Scale spaces

Motivation

Basic filters have problems to handle low SNR and textured noise. Something new is required. . .

The solution

Filtering on different scales can take noise suppression one step further.



Wavelets – the basic idea

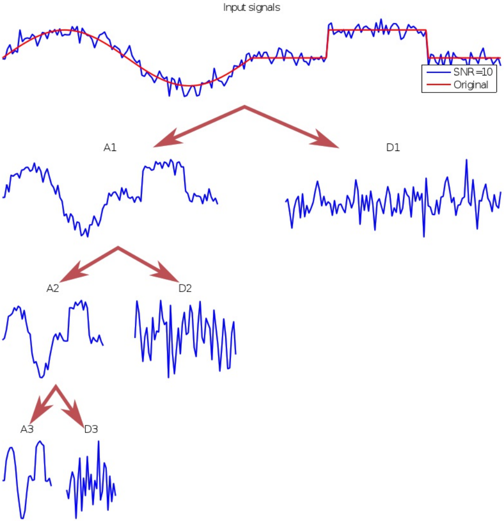
- The wavelet transform produces scales by decomposing a signal into two signals at a coarser scale containing *trend* and *details*.
- The next scale is computed using the trend of the previous transform

$$WT\{s\} \rightarrow \{a_1, d_1\}, WT\{a_1\} \rightarrow \{a_2, d_2\}, \ldots, WT\{a_{N-1}\} \rightarrow \{a_N, d_N\}$$

- The inverse transform brings s back using $\{a_N, d_1, \ldots, d_N\}$.
- Many wavelet bases exists, the choice depends on the application.
- Wavelets can have several uses:
    - Noise reduction
    - Analysis
    - Segmentation
    - Compression

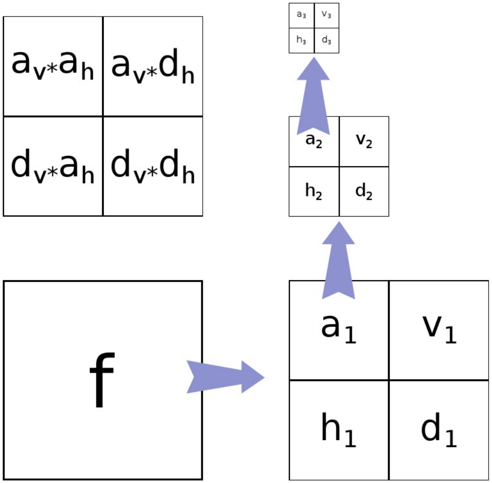
[J.C.Walker, 1999][Mallat, 2009]

Wavelet transform of a 1D signal

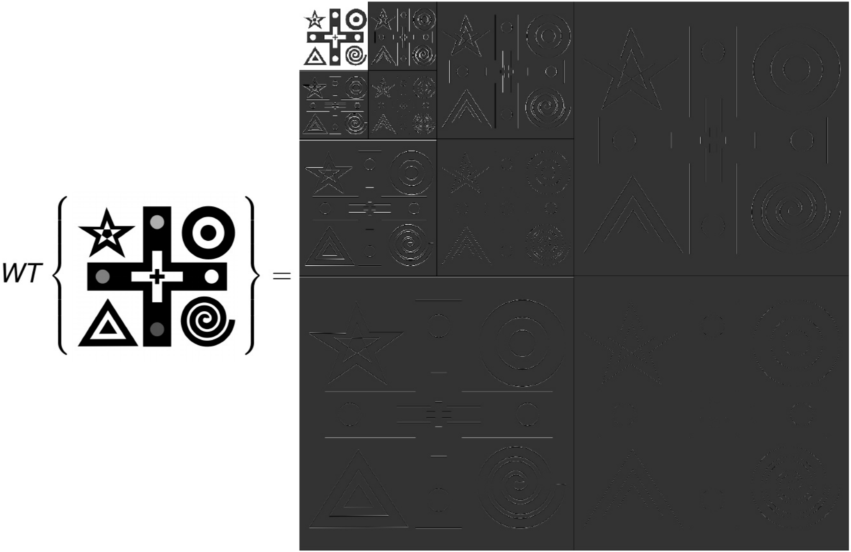


Using *symlet-4*

Wavelet transform of an image



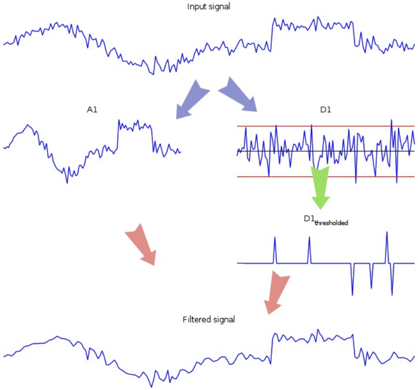
Wavelet transform of an image – example



Wavelet noise reduction

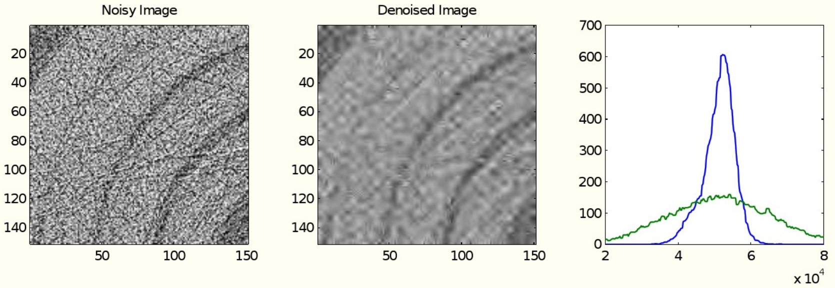
The noise is found in the detail part of the WT

- Make a WT of the signal to a level that corresponds to the scale of the unwanted information.
- Threshold the detail part $d_\gamma = |d| < \gamma\,?\,0 : d$.
- Inverse WT back to normal scale $\rightarrow$ image is filtered.



Example (Filtered using two levels of Symlet-2 wavelet)

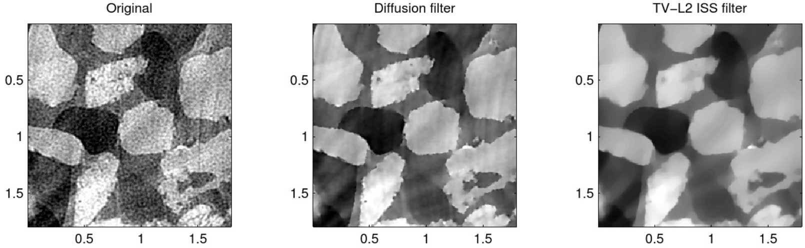
Data: Neutron CT of a lead scroll



Neutron CT of a lead scroll.

PDE based scale space filters

Filters small features faster than larger ones.



May work for applications where Linear and Rank filters fail.

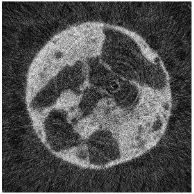
[Aubert and Kornprobst, 2002].

The starting point

The heat transport equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

- T Image to filter (intensity $\equiv$ temperature)
- κ Thermal conduction capacity



Original   Intensity diffusion

The steady state solution is a homogeneous image. . .

Controlling the diffusivity

We want to control the diffusion process. . .

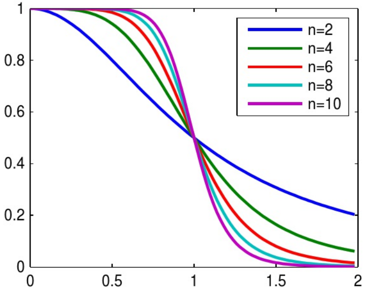
Near edges The Diffusivity $\rightarrow 0$

Flat regions The Diffusivity $\rightarrow 1$

The contrast function G is our control function

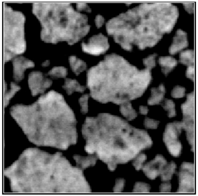
$$G(x) = \frac{1}{1 + \left(\frac{x}{\lambda}\right)^n}$$

- λ Threshold level
- n Steepness of the threshold

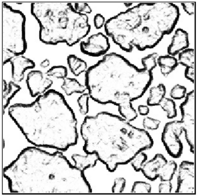


Gradient controlled diffusivity

$$\frac{\partial u}{\partial t} = G(|\nabla u|)\,\nabla^2 u$$



Image



Diffusivity map

- u Image to be filtered
- $G(\cdot)$ Non-linear function to control the diffusivity
- τ Time increment
- N Number of iterations

This filter is noise sensitive!

The non-linear diffusion filter

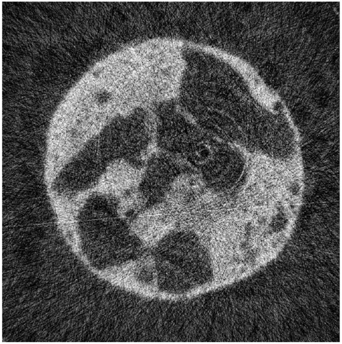
A more robust filter is obtained with

$$\frac{\partial u}{\partial t} = G(|\nabla_\sigma u|)\,\nabla^2 u \tag{4}$$

- u Image to be filtered
- $G(\cdot)$ Non-linear function to control the contrast
- τ Time increment per numerical iteration
- N Number of iterations
- ∇_σ Gradient smoothed by a Gaussian filter, width σ

Diffusion filter example

Neutron CT slice from a real-time experiment observing the coalescence of cold mixed bitumen.



Original

Iterations of non-linear diffusion

The continued development

90's During the late 90's the diffusion filter was described in terms of a regularization problem.

00's Work toward regularization of total variation minimization.

TV-L1

$$u = \underset{u \in BV(\Omega)}{\operatorname{argmin}} \left\{ \underbrace{|u|_{BV}}_{noise} + \underbrace{\frac{\lambda}{2}\|f-u\|_1}_{fidelity} \right\}$$

Rudin-Osher-Fatemi model (ROF)

$$u = \underset{u \in BV(\Omega)}{\operatorname{argmin}} \left\{ \underbrace{|u|_{BV}}_{noise} + \underbrace{\frac{\lambda}{2}\|f-u\|_2^2}_{fidelity} \right\}$$

with $|u|_{BV} = \int_\Omega |\nabla u|^2$

The inverse scale space filter

The idea

We want smooth regions with sharp edges. . .

- Turn the processing order of scale space filter upside down
- Start with an empty image
- Add large structures successively until an image with relevant features appears

The ISS filter – Some properties

- is an edge preserving filter for noise reduction.
- is defined by a partial differential equation.
- has a well defined termination point.

[Burger et al., 2006]

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The ROF filter equation

The image f is filtered by solving

$$\begin{aligned} \frac{\partial u}{\partial t} &= \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \lambda\,(f - u + v) \\ \frac{\partial v}{\partial t} &= \alpha\,(f - u) \end{aligned} \qquad (5)$$

Variables:

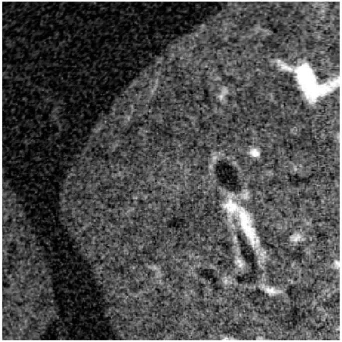
- f Input image
- u Filtered image
- v Regularization term (feedback of previous iteration)

Filter parameters

- λ Related to the scale of the features to suppress.
- α Quality refinement
- N Number of iterations
- τ Time increment

Filter iterations

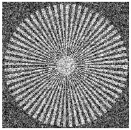
Neutron CT of dried lung filtered using 3D ISS filter



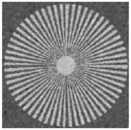
Original

Filter iterations

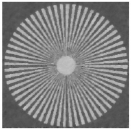
Solutions at different times



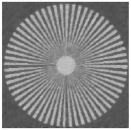
1



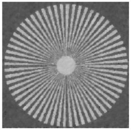
30



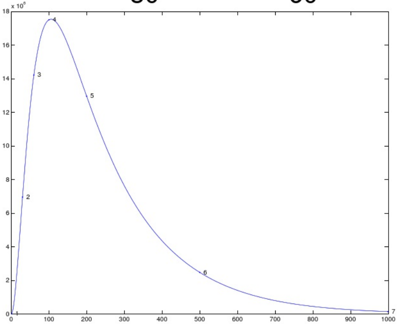
60

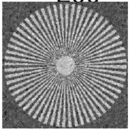


100

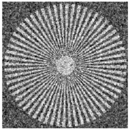


200





500



999

The idea

Smoothing normally consider information from the neighborhood like

- Local averages (convolution)
- Gradients and Curvatures (PDE filters)

Non-local smoothing average similiar intensities in a global sense.

- Every filtered pixel is a weighted average of all pixels.
- Weights computed using difference between pixel intensities.

[Buades et al., 2005]

Filter definition

The non-local means filter is defined as

$$u(p) = \frac{1}{C(p)} \sum_{q \in \Omega} v(q)\, f(p,q) \tag{6}$$

where

v and u input and result images.

$C(p)$ is the sum of all pixel weights as

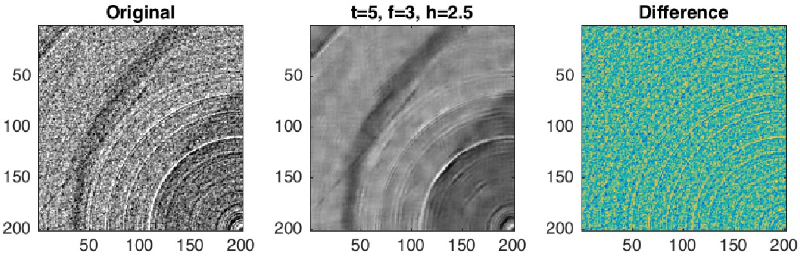
$$C(p) = \sum_{q \in \Omega} f(p,q) \tag{7}$$

$f(p,q)$ is the weighting function

$$f(p,q) = e^{-\frac{|B(q)-B(p)|^2}{h^2}} \tag{8}$$

B(x) is a neighborhood operator e.g. local average around x

Non-local means 2D – Example



Observations

- Good smoothing effect.
- Strong thin lines are preserved.
- Some patchiness related to filter parameter t, *i.e.* the size of Ω_i.

Performance complications

Problem

The orignal filter compares all pixels with all pixels...

- Complexity $\mathcal{O}(N^2)$
- Not feasible for large images, and particular 3D images!

Solution

It has been shown that not all pixels have to be compared to achieve a good filter effect.
i.e. Ω in eq 6 and 7 can be replaced by $\Omega_i << \Omega$

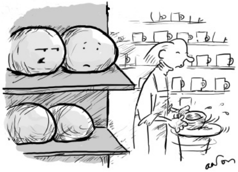
Verification

Verify the correctness of the method

"Data massage"

Filtering manipulates the data...

...avoid too strong modifications otherwise you may invent new image features!!!



Watch that man, he'll make mugs of us all!

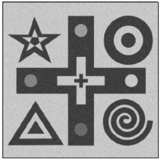
Verify the validity your method

- Visual inspection
- Difference images
- Use degraded phantom images in a "smoke test"

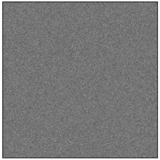
Verification using difference images

Compute pixel-wise difference between image f and g

Noisy image

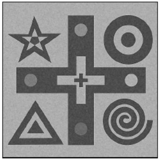


Ideal filter



Over smoothing

Intensity scaling



Geometric shift

Difference images provide first diagnosis about processing performance

Performance testing – The smoke test

- Testing term from electronic hardware testing – drive the system until something fails due to overheating...
- In general: scan the parameter space for different SNR until the method fails to identify strength and weakness of the system.

Test strategy

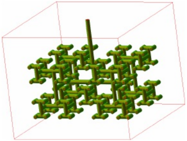
1. Create a phantom image with relevant features.
2. Add noise for different SNR to the phantom.
3. Apply the processing method with different parameters.
4. Measure the difference between processed and phantom.
5. Repeat steps 2-4 N times for better test statistics.
6. Plot the results and identify the range of SNR and parameters that produce acceptable results.

Data for evaluation

Phantom data

General purpose can be controlled

- Data with known features.
- Parameters can be changed.
  - Shape
  - Sharpness
  - Contrast
  - Noise (distribution and strength)



Labeled data

- Often 'real' data
- Labeled by experts
- Used for training and validation
  - Training of model
  - Validation
  - Test

Evaluation metrics for images

An evaluation procedure need a metric to compare the performance

Mean squared error

$$MSE(f,g) = \sum_{p\in\Omega}(f(p)-g(p))^2$$

Structural similarity index

$$SSIM(f,g) = \frac{(2\mu_f\,\mu_g + C_1)(2\sigma_{fg} + C_2)}{(\mu_f^2+\mu_g^2+C_1)(\sigma_f^2+\sigma_g^2+C_2)}$$

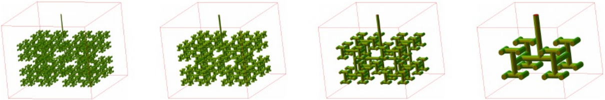
- $\mu_f,\ \mu_g$ Local mean of f and g.
- σ_{fg} Local correlation between f and g.
- $\sigma_f,\ \sigma_g$ Local standard deviation of f and g.
- $C_1,\ C_2$ Constants based on the image dynamics (small numbers).

$$MSSIM(f,g) = E[SSIM(f,g)]$$

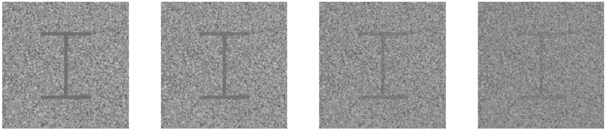
[Wang and Bovik, 2009]

Test run example

Phantom structures



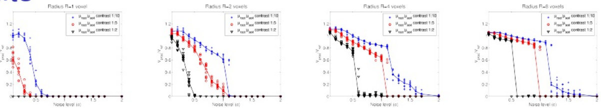
Change SNR and contrast



Process

Apply processing sequence.

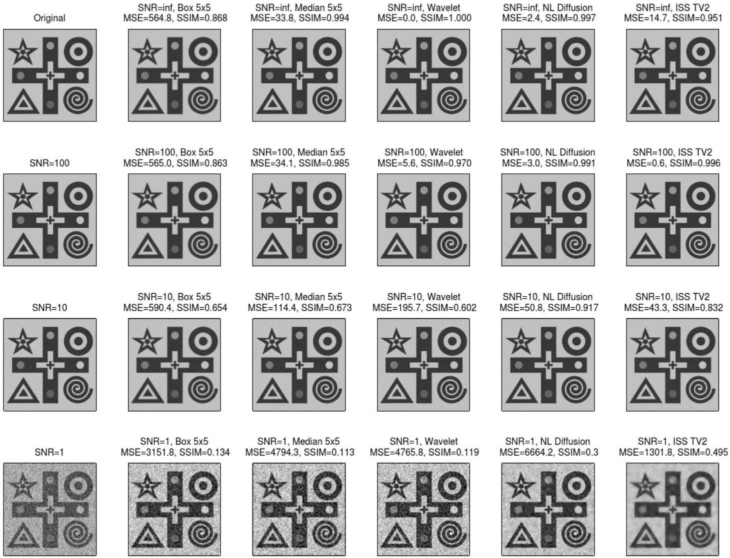
Plot results



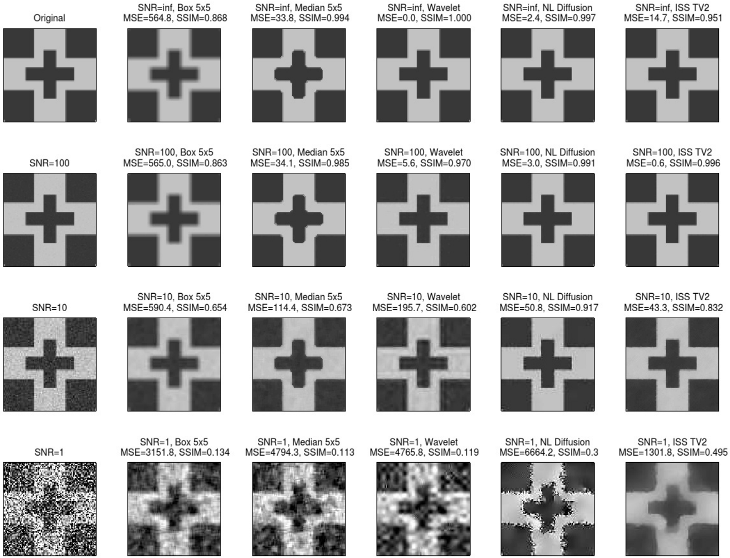
[Kaestner et al., 2006]

Summary

Many filters



Details of filter performance



Take-home message

We have looked at different ways to suppress noise and artifacts:

- Convolution
- Median filters
- Wavelet denoising
- PDE filters

Which one you select depends on

- Purpose of the data
- Quality requirements
- Available time

Remember

A good measurement is better than an enhanced bad measurement...
...but bad data can mostly be rescued if needed.

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